THE EFFECTIVE COSMOLOGICAL CONSTANT IN HIGHER ORDER GRAVITY THEORIES

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An effective time–dependent cosmological constant can be recovered for higher–order theories of gravity by extending to these ones the no–hair conjecture. The results are applied to some specific cosmological models.

Fundamental issues as the determination of gravity vacuum state, the mechanism which led the early universe to the today observed large scale structures, and the prediction of what will be the fate of the whole universe are all connected to the determination of cosmological constant. Besides, from several points of view, it seems that Einstein's theory of gravity must be generalized to higher–order or scalar–tensor theories in order to overcome shortcomings of fundamental physics and cosmology. Both topics have to be connected so that a lot of people is, recently, asking for the recovering of a sort of cosmological constant in extended theories of gravity. With this fact in mind, it is possible to enlarge the cosmic no–hair conjecture to scalar–tensor [1] and higher–order theories by taking into consideration an "effective" time–dependent cosmological constant which has to become asymptotically the "true" one yielding a de Sitter behaviour. The original no–hair conjecture claims that if there is a positive cosmological constant, all the expanding universes will approach a de Sitter behaviour. A simplified version of the conjecture can be proved [2]. It is worthwhile to note that in Wald's paper the cosmological constant is a true constant (put by hands) and the contracted Bianchi identity is not used, then the proof is independent of the evolution of matter. In order to extend no–hair conjecture to generalized theories of gravity, we have to introduce different sets of conditions (with respect to those given in [2]) since the cosmological constant is not introduced *a priori*, but it is "recovered" from dynamics of scalar fields or higher–order geometric terms in the gravitational Lagrangians. Such conditions must not use the "energy conditions", but they have to allow the introduction of a sort of effective cosmological constant which asymptotically becomes the de Sitter constant. Furthermore, it is the only term that does not

decrease with time. Hence, in an expanding universe, Λ is the asymptotically dominant term in the Einstein equations. Then, given any extended theory of gravity, it could be possible, in general, to define an effective time varying cosmological constant which becomes the "true" cosmological constant if and only if the model asymptotically approaches de Sitter (that is only asymptotically no–hair conjecture is recovered). This fact will introduce constraints on the choice of the gravitational couplings, scalar field potentials and higher–order geometrical terms which combinations can be intended as components of the effective stress–energy tensor. Here we deal with higher–order theories asking for recovering the de Sitter behaviour in the related cosmological models, using the scheme which we adopt for scalar–tensor gravity [1]. We take into account the function $f(R)$ where R is the Ricci scalar. Let us start from the action

$$\mathcal{A} = \int d^4x \sqrt{-g}\,[f(R) + \mathcal{L}_m]\ . \tag{1}$$

The standard (minimally coupled) matter gives no contribution to dymnamics when we consider the asymptotic behaviour of system. For the sake of simplicity, we discard its contribution (*i.e.* $\mathcal{L}_m = 0$) from now on. We adopt a FRW metric considering that the results can be extended to any Bianchi model [1]. The Lagrangian density in action (1) can be written as $\mathcal{L} = \mathcal{L}(a, \dot{a}, R, \dot{R})$, where a and R are canonical variables. Such a position seems arbitrary, since R is not independent of a and $\dot{a}$, but it is generally used in canonical quantization of higher–order gravitational theories [3]. In practice, the definition of R by $\ddot{a}, \dot{a}$ and a introduces a constraint which eliminates the second and higher order derivatives in time, then this last one produces a system of second order differential equations in $\{a, R\}$. In fact, using the Lagrange multiplier λ, we have that the action can be written as

$$\mathcal{A} = 2\pi^2 \int dt \left\{ f(R)a^3 - \lambda \left[R + 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \right\}\ . \tag{2}$$

The prime indicates the derivative with respect to R. In order to determine λ, we have to vary the action with respect to R, from which $\lambda = a^3 f'(R)$. Substituting into (2) and integrating by parts, we obtain the Lagrangian

$$\mathcal{L} = a^3\,[f(R) - Rf'(R)] + 6\dot{a}^2 a f'(R) + 6a^2\dot{a}\dot{R}f''(R) - 6akf'(R)\,. \tag{3}$$

Then the equations of motion are

$$\left(\frac{\ddot{a}}{a}\right) f(R)' + 2\left(\frac{\dot{a}}{a}\right) f(R)''\dot{R} + f(R)''\ddot{R} + f(R)'''\dot{R}^2 - \frac{1}{2}[Rf(R)' + f(R)] = 0\,, \tag{4}$$

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right), \tag{5}$$

which is the constraint, and

$$6\dot{a}^2 a f'(R) - a^3\left[f(R) - Rf'(R)\right] + 6a^2\dot{a}\dot{R}f''(R) + akf'(R) = 0\,. \tag{6}$$

Let us now define the auxiliary field $p \equiv f'(R)$, so that the Lagrangian (3) can be recast in the form

$$\mathcal{L} = 6a\dot{a}^2 p + 6a^2\dot{a}\dot{p} - 6akp - a^3 W(p)\,, \tag{7}$$

where

$$W(p) = h(p)p - r(p)\,, \quad r(p) = \int f'(R)dR = \int pdR = f(R)\,, \quad h(p) = R\,, \tag{8}$$

such that $h = (f')^{-1}$ is the inverse function of f'. A Lagrangian like (7) is a special kind of the so called Helmholtz Lagrangian [4]. Dynamical system becomes

$$6\left[\dot{H} + 2H^2\right] = -\frac{dW(p)}{dp}\,, \tag{9}$$

$$H^2 + H\frac{\dot{p}}{p} + \frac{W(p)}{6p} = 0\,, \tag{10}$$

$$\dot{H} = -\frac{1}{2}\left(H^2 + \frac{W(p)}{6p}\right) - \frac{1}{2}\left(\frac{\dot{p}}{p}\right)^2 - \frac{1}{2}\frac{d}{dt}\left(\frac{\dot{p}}{p}\right)\,. \tag{11}$$

Eq.(9) has the role of the Klein–Gordon equation. H is the Hubble parameter. We want to obtain an effective cosmological constant. For semplicity, we have assumed $k = 0$. Eq.(10) can be recast as

$$(H - \Lambda_{eff,\,1})(H - \Lambda_{eff,\,2}) = 0\,. \tag{12}$$

The effective cosmological constant can be formally defined as

$$\Lambda_{eff\,1,2} = -\frac{\dot{p}}{2p} \pm \sqrt{\left(\frac{\dot{p}}{2p}\right)^2 - \frac{W(p)}{6p}}\,. \tag{13}$$

We have to note that Eq.(12) defines the exact solutions $H(t) = \Lambda_{eff,\,1,2}$ which, respectively, separate the region with expanding universes ($H > 0$) from the region with contracting universes ($H < 0$) if $\rho_m \neq 0$. The effective $\Lambda_{eff,1,2}$ becomes an asymptotic constant for $t \to \infty$, if the conditions

$$\frac{\dot{p}}{p} \longrightarrow \Sigma_0\,, \quad \frac{W(p)}{6p} \longrightarrow \Sigma_1\,, \tag{14}$$

hold. From (11), we get $\dot{H} \leq 0$ if

$$H^2 \geq -\frac{W(p)}{6p}\,. \tag{15}$$

Conditions (14) gives the asymptotic behaviour of field p and potential $W(p)$. By a little algebra, we obtain that asymptotically must be

$$\Sigma_0 = 0\,, \qquad f(R) = f_0(R + 6\Sigma_1)\,; \tag{16}$$

where f_0 is an arbitrary constant. The asymptotic solution is then

$$H^2 = \Sigma_1\,, \qquad p = p_0\,, \qquad \dot{H} = 0\,. \tag{17}$$

From Eq.(9), or, equivalently, from the constraint (5), we get

$$R = -12H^2 = -12\Sigma_1\,. \tag{18}$$

The no–hair theorem is restored without using Bianchi identities (*i.e.* the Klein–Gordon equation). The de Sitter solution of the Einstein gravity is exactly recovered if $\Sigma_1 = \Lambda/3$. It depends on the free constant f_0 in (16) which is assigned by introducing ordinary matter in the theory. This means that, asymptotically,

$$f(R) = f_0(R + 2\Lambda)\,. \tag{19}$$

The situation is not completely analogue to the scalar–tensor case [1] since the request that asymptotically $a(t) \rightarrow \exp(\Lambda t)$, univocally "fixes" the asymptotic form of $f(R)$. Inversely, any fourth–order theory which asymptotically has de Sitter solutions, has to assume the form (19). We have to stress the fact that it is the *a priori* freedom in choosing $f(R)$ which allows to recover an asymptotic cosmological constant (which is not present in the trivial case $f(R) = R$, unless it is put by hand) so that de Sitter solution is, in some sense, intrinsic in higher–order theories. A pure higher than fourth–order gravity theory is recovered, for example, with the choice

$$\mathcal{A} = \int d^4x\sqrt{-g}f(R, \Box R)\,. \tag{20}$$

which is, in general, an eighth–order theory. If f depends only linearly on $\Box R$, we have a sixth–order theory. As above, we can get a FRW pointlike Lagrangian with the position $\mathcal{L} = \mathcal{L}(a, \dot{a}, R, \dot{R}, \Box R, (\dot{\Box R}))$. Also here, we consider R and $\Box R$ as two independent fields and use the method of Lagrange

multipliers to eliminate higher derivatives than one in time. We obtain the Helmholtz–like Lagrangian

$$\mathcal{L} = a^3\left[f - R\mathcal{G} + 6H^2\mathcal{G} + 6H\dot{\mathcal{G}} - \frac{6k}{a^2}\mathcal{G} - \Box R\mathcal{F} - \dot{R}\dot{\mathcal{F}}\right], \tag{21}$$

where $\mathcal{G} = \frac{\partial f}{\partial R} + \Box\frac{\partial f}{\partial \Box R}$ and $\mathcal{F} = \frac{\partial f}{\partial \Box R}$. The equations of motion are

$$H^2 + H\left(\frac{\dot{\mathcal{G}}}{\mathcal{G}}\right) + \frac{\chi}{6\mathcal{G}} = 0\,; \tag{22}$$

$$\dot{H} = -\frac{1}{2}\left(H^2 + \frac{\chi}{6\mathcal{G}}\right) - \frac{1}{2}\left(\frac{\dot{\mathcal{G}}}{\mathcal{G}}\right)^2 - \frac{1}{2}\frac{d}{dt}\left(\frac{\dot{\mathcal{G}}}{\mathcal{G}}\right) - -\frac{\dot{R}\dot{\mathcal{F}}}{2\mathcal{G}}\,, \tag{23}$$

$$R = -6[\dot{H} + 2H^2]\,, \quad \Box R = \ddot{R} + 3H\dot{R}\,, \tag{24}$$

where Eqs.(24) have the role of Klein–Gordon equations for the fields R and $\Box R$ and are also "constraints" for such fields. The quantity χ is $\chi = R\mathcal{G} + \mathcal{F}\Box R - \dot{R}\dot{\mathcal{F}} - f$. We can define an effective cosmological constant as

$$\Lambda_{eff,\,1,2} = -\frac{\dot{\mathcal{G}}}{2\mathcal{G}} \pm \sqrt{\left(\frac{\dot{\mathcal{G}}}{2\mathcal{G}}\right)^2 - \frac{\chi}{6\mathcal{G}}}\,. \tag{25}$$

$\Lambda_{eff,\,1,2}$ become asymptotically constants if

$$\frac{\dot{\mathcal{G}}}{\mathcal{G}} \longrightarrow \Sigma_0\,, \quad \frac{\chi}{6\mathcal{G}} \longrightarrow \Sigma_1\,. \tag{26}$$

From (23), we have $\dot{H} \leq 0$ if

$$H^2 \geq -\frac{\chi}{6\mathcal{G}}\,, \quad \frac{\dot{R}\dot{\mathcal{F}}}{\mathcal{G}} \geq 0\,. \tag{27}$$

The quantities χ, $\mathcal{G}$, and Λ_{eff} are functions of two fields and the de Sitter asymptotic regime selects particular surfaces $\{R, \Box R\}$. In conclusion, if we compare the conditions in [2] with ours, we have that

$$\begin{array}{ccc} \text{(no–hair)} & & \text{(our asymptotic conditions)} \\ \left(H - \sqrt{\frac{\Lambda}{3}}\right)\left(H + \sqrt{\frac{\Lambda}{3}}\right) \geq 0 & \Longleftrightarrow & (H-\Lambda_1)(H-\Lambda_2) \geq 0, \\ \dot{H} \leq \frac{\Lambda}{3} - H^2 \leq 0 & \Longrightarrow & \dot{H} \leq 0\,. \end{array}$$

and in this sense the no–hair theorem is extended. The above discussion can be realized in specific cosmological models. As in [1] for scalar–tensor theories, it is possible to give examples where, by fixing the higher–order theory, the asymptotic de Sitter regime is restored in the framework of no–hair theorem. The presence of standard fluid matter can be implemented by adding the term $\mathcal{L}_m = Da^{3(1-\gamma)}$ into the FRW–pointlike Lagrangian. It is a sort of pressure term. For our purposes it is not particularly relevant. The conditions for the existence and stability of de Sitter solutions for fourth–order theories $f(R)$ are widely discussed [5]. In particular, it is shown that, for R covariantly constant (*i.e.* $R = R_0$), as recovered in our case for $R \to -12\Sigma_1$, the field equations yield the existence condition $R_0 f'(R_0) = 2f(R_0)$. Thus, given any $f(R)$ theory, if there exists a solution R_0, the theory contains a de Sitter solution. From our point of view, any time that the ratio $\dot{f}(R(t))/f(R(t))$ converges to a constant, a de Sitter (asymptotic) solution exists. On the other hand, given, for example, a theory of the form $f(R) = \Sigma_{n=0}^{N} a_n R^n$, the above condition is satisfied if the polynomial equation $\Sigma_{n=0}^{N}(2-n)a_n R_0^n = 0$, has real solutions. Examples of de Sitter asymptotic behaviours recovered in this kind of theories are given in literature[6]. Examples of theories higher than fourth–order in which asymptotic de Sitter solutions are recovered are discussed in [7]. There is discussed under which circumstances the de Sitter space–time is an attractor solution in the set of spatially flat FRW models. Several results are found: for example, a R^2 non–vanishing term is necessarily required (*i.e.* a fourth–order term cannot be escaped); the models are independent of dimensionality of the theory; more than one inflationary phase can be recovered. Reversing the argument from our point of view, a wide class of cosmological models coming from higher–order theories, allows to recover an asymptotic cosmological constant which seems an intrinsic feature if Einstein–Hilbert gravitational action is modified by higher–order terms. In this sense, and with the conditions given above, the cosmological no–hair theorem is extended.

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